

POSITIVITY OF SIMPLICIAL VOLUME FOR NONPOSITIVELY CURVED MANIFOLDS WITH A RICCI-TYPE CURVATURE CONDITION

CHRIS CONNELL[†] AND SHI WANG

ABSTRACT. We show that closed manifolds supporting a nonpositively curved metric with negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature, have positive simplicial volume. This answers a special case of a conjecture of Gromov.

1. INTRODUCTION

In the early 80's, Gromov ([Gro82]) and Thurston ([Thu77, Chapter 6]) introduced the notion of *simplicial volume* for any closed, connected and orientable manifold M . The simplicial volume, denoted $\|M\|$, is a nonnegative real number which measures how efficiently the fundamental class of M can be represented by real singular cycles. It is defined to be,

$$\|M\| = \inf \left\{ \sum_i |a_i| : \left[\sum_i a_i \sigma_i \right] = [M] \in H_n(M, \mathbb{R}) \right\},$$

where the infimum is taken over all singular cycles with real coefficients representing the fundamental class in the top homology group of M .

One reason for the importance of this invariant stems from its sitting at the bottom of a chain of inequalities relating it to other invariants defined in terms of geometric and dynamical properties of the manifold. Thus when it does not vanish, there are numerous implications beyond the topology of M .

The simplicial volume has been shown to vanish for several large classes of manifolds. These include those admitting a nondegenerate circle action, or more generally a polarized \mathcal{F} -structure [Fuk87, CG90, Gro82], certain affine manifolds ([BCL16]), and manifolds with amenable fundamental group ([Gro82]). When nonzero, the exact value of the simplicial volume has been calculated only for a few cases including hyperbolic manifolds [Thu77, Gro82], products of surfaces ([BK08]) and Hilbert modular surfaces ([LS09b]). More broadly, positivity of simplicial volume has been established for only a few special classes of manifolds including higher genus surface-by-surface bundles ([HK01]), negatively curved manifolds ([Gro82]), certain generalized graph manifolds ([CS12]), and most closed locally symmetric spaces of higher rank and some noncompact finite volume ones as well

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([LS06, BK07, KK12, LS09b]). Some connections to other invariants have been established (see e.g. [Kue04, KK15, LS09a, LP16, Buc09, BBF⁺14, BKK14]), though calculations remain difficult.

The situation for admitting a nonpositively curved metric remains delicate. Some of these fall under the scope of the afore-mentioned results (e.g. admitting polarized \mathcal{F} -structures, certain affine structures, certain graph structures, higher rank locally symmetric structures, etc...), and as such vanishing or positivity of the simplicial volume is known. Among the remainder, the following conjecture has been attributed to Gromov ([Sav82] and see also [Gro82, p.11]):

Conjecture 1. *Any closed manifold of nonpositive curvature and negative Ricci curvature has $\|M\| > 0$.*

Remark 1.1. Note that the nonpositive curvature assumption is necessary here as the 3-sphere admits a metric of negative Ricci curvature ([Bro89]), and yet has vanishing simplicial volume.

Nonpositively curved Riemannian manifolds can be classified by their geometric rank, which is the minimum dimension of parallel Jacobi fields along geodesics. The higher rank manifolds turn out to have universal covers which are either metric products or symmetric spaces of noncompact type ([BBS85, BS87]), and hence their simplicial volume is already understood. The remaining class of geometric rank one manifolds is somewhat mysterious and includes manifolds with both vanishing and nonvanishing simplicial volume.

In this paper, we consider the simplicial volume of arbitrary closed manifolds admitting a nonpositively curved metric with an additional negative Ricci-type bound. More specifically, we establish a special case and weakened form of Gromov's conjecture under the condition that we replace $\text{Ric} < 0$ with $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} < 0$ where,

Definition 1. For $u, v \in T_x M$,

$$\text{Ric}_k(u, v) = \sup_{\substack{V \subset T_x M \\ \dim V = k}} \text{Tr } R(u, \cdot, v, \cdot)|_V,$$

where $R(u, \cdot, v, \cdot)|_V$ is the restriction of the curvature tensor to $V \times V$, and thus the trace is with respect to any orthonormal basis of V . Lastly, we set $\text{Ric}_k = \sup_{v \in T^1 M} \text{Ric}_k(v, v)$.

Namely, we show:

Theorem 1. *Let M be an oriented closed manifold of dimension n admitting a Riemannian metric of nonpositive curvature with $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} < 0$. Then the simplicial volume satisfies $\|M\| > 0$.*

Remark 1.2. Note that the Ricci condition is necessary both in the conjecture and in Theorem 1 since any manifold that splits as a product with S^1 ,

e.g. $M = N \times S^1$ for a nonpositively curved manifold N is nonpositively curved but has $\|M\| = 0$. More generally, any manifold with a polarized \mathcal{F} -structure has $\|M\| = 0$ (see [CG90]). There are many manifolds of non-positive curvature admitting such an \mathcal{F} -structure, e.g. the famous twisted product of surface by circle examples (see [CG86]).

Also, for any metric on M with $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} < 0$ we may compute an explicit bound for the simplicial volume in terms of the dimension n and its curvature tensor (up to second covariant derivatives). In principal a bound for the simplicial volume can be obtained by taking the supremum of this bound over all metrics, but we have not attempted to estimate this.

Remark 1.3. Examples of spaces admitting $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} < 0$ can be constructed by deforming negatively curved manifolds, e.g. in the neighborhood of a closed geodesic for instance such that a few tangent directions have vanishing curvature with the remainder being negative.

More interesting, and quite general, examples which do not admit negatively curved metrics, can be constructed as follows. Start with any closed manifold N of nonpositive curvature and dimension k . Using the main result of [Tam13], we obtain a $k + 1$ -manifold N_1 containing N as a totally geodesic submanifold and whose only planes of 0-curvature are tangent to $N \subset N_1$. (This latter fact was not explicitly stated in her theorem, but follows from the strict hyperbolization and warped product construction used in its proof.) Iterating this construction we produce a sequence N_1, N_2, \dots manifolds with totally geodesic embeddings $N_i \subset N_{i+1}$. For some $j \leq 3k$ this process yields a manifold $M = N_j$ of dimension $n = k + j$ which has $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} < 0$, as desired.

The simplest nontrivial example starts with N a flat 2-torus and iterates six times to produce a nonpositively curved 8-manifold M with negative Ric_3 -curvature and a single 2-flat which is isolated in the sense of [HK05].

Note that the existence of a closed flat in M for some nonpositively curved metric implies the existence of a \mathbb{Z}^2 subgroup in $\pi_1(M)$, which in turn implies that $\pi_1(M)$ is not Gromov hyperbolic. In particular, such a smooth manifold M , including many of these examples, cannot admit any negatively curved metric. (In fact, the existence of any flat in a closed nonpositively curved manifold is conjectured to imply the existence of a closed flat.)

Other interesting cases include starting with N a closed locally symmetric space of higher rank. Combining this construction with gluing techniques across negatively curved portions of each N_i , we can construct many other types of interesting examples to which our theorems apply.

We prove our first result using estimates for the Jacobian of the barycenter maps. Using these we can make an extension to the case of maps.

Theorem 2. *Given any closed oriented Riemannian manifolds N and M of dimension n , and assume M is nonpositively curved and has $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} < 0$. There is a constant $C > 0$, depending only on the metric of \widetilde{M} and scale*

invariant, such that if $f : N \rightarrow M$ is any continuous map, then

$$h(N)^n \text{Vol}(N) \geq C |\deg f| h(M)^n \text{Vol}(M)$$

where $h(N)$ is the volume growth entropy of N .

Remark 1.4. We note that in the theorem we also have $h(N)^n \text{Vol}(N) \geq \sup_{f,M} \{C |\deg f| h(M)^n \text{Vol}(M)\}$ where the supremum is taken over all M satisfying the hypotheses and maps $f : N \rightarrow M$. Since this supremum is some positive universal constant, we may take C in the theorem to be a universal constant, provided there is not a sequence of Riemannian manifolds $\{M_i\}$ satisfying the hypotheses and maps $f_i : N \rightarrow M_i$, from some N , such that $|\deg f_i| h(M_i)^n \text{Vol}(M_i) \rightarrow \infty$. While there are examples of nonpositively curved graph manifolds for which this happens, we are not aware of any such examples satisfying our hypotheses.

Also, the assumption that N and M be oriented can be removed provided we use the appropriate generalization of $\deg(f)$ (see e.g. [Olu53]).

Finally, in Section 7 we will present some direct corollaries and applications of these theorems to bounded cohomology, the co-Hopf property of groups and positivity of some related topological invariants.

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2. STRAIGHTENING METHOD

To prove the positivity of simplicial volume of certain manifolds, we follow the general approach of Thurston [Thu77]. The essential idea is that if a manifold admits a straightening, then its simplicial volume is positive. For simplicity, we follow the notion of [LS06] as a brief summary of Thurston's approach.

Definition 2. ([LS06]) Let \widetilde{M}^n be the universal cover of an n -dimensional manifold M^n . We denote by Γ the fundamental group of M^n , and by $C_*(\widetilde{M}^n)$ the real singular chain complex of \widetilde{M}^n . Equivalently, $C_k(\widetilde{M}^n)$ is the free \mathbb{R} -module generated by $C^0(\Delta^k, \widetilde{M}^n)$, the set of singular k -simplices in \widetilde{M}^n , where Δ^k is equipped with some fixed Riemannian metric. We say a collection of maps $st_k : C^0(\Delta^k, \widetilde{M}^n) \rightarrow C^0(\Delta^k, \widetilde{M}^n)$ is a straightening if it satisfies the following conditions:

- (1) the maps st_k are Γ -equivariant,
- (2) the maps st_* induce a chain map $st_* : C_*(\widetilde{M}^n, \mathbb{R}) \rightarrow C_*(\widetilde{M}^n, \mathbb{R})$ that is Γ -equivariantly chain homotopic to the identity,
- (3) the image of st_n lies in $C^1(\Delta^n, \widetilde{M}^n)$, that is, the top dimensional straightened simplices are C^1 ,
- (4) there exists a constant C depending on \widetilde{M}^n and the chosen Riemannian metric on Δ^n , such that for any $f \in C^0(\Delta^n, \widetilde{M}^n)$, and

corresponding straightened simplex $st_n(f) : \Delta^n \rightarrow \widetilde{M}^n$, there is a uniform upper bound on the Jacobian of $st_n(f)$:

$$|\text{Jac}(st_n(f))(\delta)| \leq C$$

for all $\delta \in \Delta^n$.

Theorem 3. ([Thu77][LS06]) *If \widetilde{M}^n admits a straightening, then the simplicial volume of M is positive.*

Therefore, the question can be solved if we can find a straightening. It is worth noting that in the above theorem, one need only a weaker condition than (4) of Definition 2: that there is a uniform upper bound on the volume—the integral of Jacobian among all top dimensional straightened simplices. In fact, for a non-positively curved manifold, it is not difficult to find a collection of maps that satisfies only (1), (2) and (3) of Definition 2. For instance, given any simplex, we can inductively take the geodesic coning of its ordered vertices, but it is not so clear whether the straightened simplices have uniformly bounded Jacobian or volume. However, in the context of locally symmetric spaces of non-compact type excluding \mathbb{H}^2 , and $\text{SL}(3, \mathbb{R})/\text{SO}(3)$ (See also [BK07] for another type of straightening in the case of $\text{SL}(3, \mathbb{R})/\text{SO}(3)$), Lafont and Schmidt [LS06] introduced the barycentric straightening to show the positivity of simplicial volume, where the Jacobian estimate as in (4) of Definition 2 relies on previous work of Connell and Farb [CF03b, CF03a, CF17].

In this article, we will apply the barycentric straightening to geometric rank one manifolds. In higher rank symmetric spaces $X = G/K$ the Patterson-Sullivan measures are K -invariant and are nicely supported on the Furstenberg boundary of X , the Jacobian estimate then turns into a careful analysis on certain Lie groups and Lie algebras, while losing symmetries in geometric rank one, and lacking tools in Lie theory, we can not transplant the estimate to the situation we have. Instead we will relate the Hessian of Busemann functions with sectional curvatures to make the eigenvalue matching. Our estimate allows some presence of zero curvatures of the manifold, as long as they occur in a small dimensional subspace, that is, for any direction v in the tangent bundle, we allow up to $\lfloor \frac{n}{4} \rfloor + 1$ many directions (including v itself) to have zero curvatures with v . We will show that, for manifolds satisfying our *negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature condition*, the barycentric straightening is a straightening in the sense of Definition 2. As a result, all closed nonpositively curved manifolds with negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature have positive simplicial volume.

3. PATTERSON-SULLIVAN MEASURES AND BARYCENTERS

We will briefly discuss in this section the Patterson-Sullivan measures in geometric rank one spaces, following the work of Knieper [Kni98]. See also the original Patterson-Sullivan theory for Fuchian groups [Pat76], and in higher rank symmetric spaces [Alb99].

We fix the notion and let M be a compact nonpositively curved geometric rank one manifold, \widetilde{M} the universal cover of M , and Γ the fundamental group of M . In [Kni98], Knieper showed that there exists a unique family of finite Borel measures $\{\mu_x\}_{x \in \widetilde{M}}$ fully supported on $\partial\widetilde{M}$, called the Patterson-Sullivan measures, which satisfies:

- (1) μ_x is Γ -invariant, for all $x \in \widetilde{M}$,
- (2) $\frac{d\mu_x}{d\mu_y}(\theta) = e^{hB(x,y,\theta)}$, for all $x, y \in \widetilde{M}$, and $\theta \in \partial\widetilde{M}$,

where h is the volume entropy of M , and $B(x, y, \theta)$ is the Busemann function of \widetilde{M} . Recall that, the Busemann function B is defined by

$$B(x, y, \theta) = \lim_{t \rightarrow \infty} (d_{\widetilde{M}}(y, \gamma_\theta(t)) - t)$$

where $\gamma_\theta(t)$ is the unique geodesic ray from x to θ .

We fix a basepoint O in \widetilde{M} and shorten $B(O, y, \theta)$ to just $B(y, \theta)$. We note that for fixed $\theta \in \partial\widetilde{M}$ the Busemann function $B(x, \theta)$ is convex on \widetilde{M} , and the nullspace of its Hessian $DdB_{(x,\theta)}$ in direction $v_{x\theta}$ —connecting x to θ have zero sectional curvature with $v_{x\theta}$ (the inverse is not true). Furthermore, if \widetilde{M} is assumed to be Ricci negative, and if ν is any finite Borel measure fully supported on $\partial\widetilde{M}$, by taking the integral of $B(x, \theta)$ with respect to ν , we obtain a strictly convex function (see the lemma below)

$$x \mapsto \mathcal{B}_\nu(x) := \int_{\partial\widetilde{M}} B(x, \theta) d\nu(\theta)$$

Hence we can define the barycenter $\text{bar}(\nu)$ of ν to be the unique point in \widetilde{M} where the function attains its minimum. It is clear that this definition is independent of the choice of basepoint O . The following lemma shows why the above function is strictly convex.

Lemma 3.1. *Following the above notion, if we assume \widetilde{M} have strictly negative Ricci curvature, and ν be any finite Borel measure that is fully supported on $\partial\widetilde{M}$, then the function*

$$x \mapsto \mathcal{B}_\nu(x)$$

is strictly convex.

Proof. It is equivalent to show that the Hessian

$$\int_{\partial\widetilde{M}} DdB_{(x,\theta)}(\cdot, \cdot) d\nu(\theta)$$

is positive definite. To see this, let $u \in T_x^1\widetilde{M}$ be an arbitrary unit vector, we claim there exists $\theta_0 \in \partial\widetilde{M}$ such that $v_{x\theta_0}$ is orthogonal to u , and $DdB_{(x,\theta_0)}(u, u) > 0$. If not, $DdB_{(x,\theta)}(u, u) = 0$ for all θ with $v_{x\theta}$ orthogonal to u , this implies by comparison theorem that the sectional curvatures of the two planes spanned by $v_{x\theta}$ and u are all 0 (See also Theorem 4 for an explicit estimate), which contradicts with the fact that the Ricci curvature in direction u is strictly negative. Therefore, there exists $\theta_0 \in \partial\widetilde{M}$ so that

$DdB_{(x,\theta_0)}(u, u) = \delta_0 > 0$. By continuity, there exists an open neighborhood U of θ_0 , such that $DdB_{(x,\theta)}(u, u) > \delta_0/2$ for all $\theta \in U$. We hence have

$$\int_{\widetilde{\partial M}} DdB_{(x,\theta)}(u, u) d\nu(\theta) \geq \int_U DdB_{(x,\theta)}(u, u) d\nu(\theta) > \delta_0/2 \cdot \nu(U) > 0$$

since ν has full support in $\widetilde{\partial M}$. Note that u is selected arbitrarily, so the Hessian

$$\int_{\widetilde{\partial M}} DdB_{(x,\theta)}(\cdot, \cdot) d\nu(\theta)$$

is positive definite, and the function

$$x \mapsto \mathcal{B}_\nu(x)$$

is strictly convex. □

4. BARYCENTRIC STRAIGHTENING

We now discuss the barycentric straightening introduced by Lafont and Schmidt [LS06] (based on the barycenter method originally developed by Besson, Courtois, and Gallot [BCG95]). We follow the notion in the above section, and we further denote by Δ_s^k the standard spherical k -simplex in the Euclidean space, that is

$$\Delta_s^k = \left\{ (a_1, \dots, a_{k+1}) \mid a_i \geq 0, \sum_{i=1}^{k+1} a_i^2 = 1 \right\} \subseteq \mathbb{R}^{k+1},$$

with the induced Riemannian metric from \mathbb{R}^{k+1} , and with ordered vertices (e_1, \dots, e_{k+1}) . Given any singular k -simplex $f : \Delta_s^k \rightarrow \widetilde{M}$, with ordered vertices $V = (x_1, \dots, x_{k+1}) = (f(e_1), \dots, f(e_{k+1}))$, we define the k -straightened simplex

$$st_k(f) : \Delta_s^k \rightarrow \widetilde{M}$$

$$st_k(f)(a_1, \dots, a_{k+1}) := \text{bar} \left(\sum_{i=1}^{k+1} a_i^2 \nu_{x_i} \right)$$

where $\nu_{x_i} = \mu_{x_i} / \|\mu_{x_i}\|$ is the normalized Patterson-Sullivan measure at x_i . We notice that $st_k(f)$ is determined by the (ordered) vertex set V , and we denote $st_k(f)(\delta)$ by $st_V(\delta)$, for $\delta \in \Delta_s^k$.

Problem 1. Is barycentric straightening a straightening in the sense of Definition 2?

The answer to this question is not known in general. Actually it is not difficult to see the barycentric straightening satisfies (1)-(3) of Definition 2, and the proof is similar to that of [LS06, Property (1)-(3)]. Note that to check (3), the C^1 smooth of top dimensional straightened simplices, we need the result of Lemma 3.1 in order to apply the inverse function theorem.

To check (4) that the Jacobian of top dimensional straightened simplices are uniformly bounded, we estimate as follows (which is also similar to [LS06,

Property (4)]). For any $\delta = \sum_{i=1}^{n+1} a_i e_i \in \Delta_s^n$, $st_n(f)(\delta)$ is defined to be the unique point where the function

$$x \mapsto \int_{\partial \widetilde{M}} B(x, \theta) d \left(\sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right) (\theta)$$

is minimized. Hence, by differentiating at that point, we get the 1-form equation

$$\int_{\partial \widetilde{M}} dB_{(st_V(\delta), \theta)}(\cdot) d \left(\sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right) (\theta) \equiv 0$$

which holds identically on the tangent space $T_{st_V(\delta)} \widetilde{M}$. Differentiating in a direction $u \in T_\delta(\Delta_s^n)$ in the source, one obtains the 2-form equation

$$(4.1) \quad \begin{aligned} & \sum_{i=1}^{n+1} 2a_i \langle u, e_i \rangle_\delta \int_{\partial \widetilde{M}} dB_{(st_V(\delta), \theta)}(v) d\nu_{x_i}(\theta) \\ & + \int_{\partial \widetilde{M}} DdB_{(st_V(\delta), \theta)}(D_\delta(st_V)(u), v) d \left(\sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right) (\theta) \equiv 0 \end{aligned}$$

which holds for every $u \in T_\delta(\Delta_s^n)$ and $v \in T_{st_V(\delta)}(\widetilde{M})$.

Now we define two positive semidefinite symmetric endomorphisms H_δ and K_δ on $T_{st_V(\delta)}(\widetilde{M})$:

$$\begin{aligned} \langle H_\delta(v), v \rangle_{st_V(\delta)} &= \int_{\partial \widetilde{M}} dB_{(st_V(\delta), \theta)}^2(v) d \left(\sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right) (\theta) \\ \langle K_\delta(v), v \rangle_{st_V(\delta)} &= \int_{\partial \widetilde{M}} DdB_{(st_V(\delta), \theta)}(v, v) d \left(\sum_{i=1}^{n+1} a_i^2 \nu_{x_i} \right) (\theta) \end{aligned}$$

Note from Lemma 3.1 that K_δ is positive definite. From Equation (4.1), we obtain, for $u \in T_\delta(\Delta_s^n)$ a unit vector and $v \in T_{st_V(\delta)}(\widetilde{M})$ arbitrary, the following

$$(4.2) \quad \begin{aligned} |\langle K_\delta((D_\delta(st_V)(u)), v) \rangle| &= \left| - \sum_{i=1}^{n+1} 2a_i \langle u, e_i \rangle_\delta \int_{\partial \widetilde{M}} dB_{(st_V(\delta), \theta)}(v) d\nu_{x_i}(\theta) \right| \\ &\leq \left(\sum_{i=1}^{n+1} \langle u, e_i \rangle_\delta^2 \right)^{1/2} \left(\sum_{i=1}^{n+1} 4a_i^2 \left(\int_{\partial \widetilde{M}} dB_{(st_V(\delta), \theta)}(v) d\nu_{x_i}(\theta) \right)^2 \right)^{1/2} \\ &\leq 2 \left(\sum_{i=1}^{n+1} a_i^2 \int_{\partial \widetilde{M}} dB_{(st_V(\delta), \theta)}^2(v) d\nu_{x_i}(\theta) \int_{\partial \widetilde{M}} 1 d\nu_{x_i} \right)^{1/2} \\ &= 2 \langle H_\delta(v), v \rangle^{1/2} \end{aligned}$$

via two applications of the Cauchy-Schwartz inequality.

For points $\delta \in \Delta_s^n$ where st_V is nondegenerate, we now pick orthonormal bases $\{u_1, \dots, u_n\}$ on $T_\delta(\Delta_s^n)$, and $\{v_1, \dots, v_n\}$ on $S \subseteq T_{st_V(\delta)}(\widetilde{M})$. We choose these so that $\{v_i\}_{i=1}^n$ are eigenvectors of H_δ , and $\{u_1, \dots, u_n\}$ is the resulting basis obtained by applying the orthonormalization process to the collection of pullback vectors $\{(K_\delta \circ D_\delta(st_V))^{-1}(v_i)\}_{i=1}^n$. By the choice of bases, the matrix $(\langle K_\delta \circ D_\delta(st_V)(u_i), v_j \rangle)$ is upper triangular, so we obtain

$$\begin{aligned} |\det(K_\delta) \cdot \text{Jac}_\delta(st_V)| &= |\det(\langle K_\delta \circ D_\delta(st_V)(u_i), v_j \rangle)| \\ &= \left| \prod_{i=1}^n \langle K_\delta \circ D_\delta(st_V)(u_i), v_i \rangle \right| \\ &\leq \prod_{i=1}^n 2 \langle H_\delta(v_i), v_i \rangle^{1/2} \\ &= 2^n \det(H_\delta)^{1/2} \end{aligned}$$

where the middle inequality is obtained via Equation (4.2). Hence we get the inequality

$$|\text{Jac}_\delta(st_V)| \leq 2^n \cdot \frac{\det(H_\delta)^{1/2}}{\det(K_\delta)}$$

In order to obtain uniform bounds on the Jacobian, we need a similar “BCG type” of estimate to bound $\det(H_\delta)^{1/2}/\det(K_\delta)$ in the context of geometric rank one spaces. Essentially, what lies behind is the question of eigenvalue matching, that is, with small eigenvalues of K_δ can we find enough eigenvalues of H_δ to cancel? The main difficulty of the question in general occurs when the space admits large flats, which will result in too many small eigenvalues of K_δ . However, under the assumption that the manifold has negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature, we can make the cancellation and hence obtain a uniform bound on the Jacobian. We will establish this in later sections (See Theorem 5), and therefore we summarize the discussion above into the following proposition.

Proposition 4.1. *Let K_δ, H_δ be the two positive semi-definite symmetric forms defined as above (note K_δ is actually positive definite). Assume there exists a constant C that only depends on \widetilde{M} , with the property that*

$$\frac{\det(H_\delta)^{1/2}}{\det(K_\delta)} \leq C$$

Then the quantity $|\text{Jac}(st_V)(\delta)|$ is universally bounded – independent of the choice of $(n+1)$ -tuple of points $V \subset \widetilde{M}$, and of the point $\delta \in \Delta_s^n$. Hence the barycentric straightening is a straightening.

5. ESTIMATING THE JACOBIAN

In this section, we establish Theorem 5 as our key estimate towards both Theorem 1 and 2, and consequently we prove Theorem 1. We start by introducing and analyzing our negative Ricci curvature condition.

Definition 3. For any positive semi-definite linear endomorphism $A : V^n \rightarrow V^n$, and for any $k = 1, 2, \dots, n$, we define the k -th trace of A , denoted by $\text{Tr}_k(A)$, to be

$$\inf_{V_k \subset V^n} \text{Tr}(A|_{V_k}),$$

where V_k is a k -dimensional subspace (not necessarily invariant under A) of V^n , and A is viewed as a bilinear form when taking restrictions. Equivalently, it is the sum of k least eigenvalues of A .

Proposition 5.1. *Let X be a compact manifold, and μ a regular probability measure on X . If A maps X continuously into the set of all $n \times n$ positive semi-definite matrices, then*

$$\text{Tr}_k\left(\int_X A(x) d\mu(x)\right) \geq \inf_{x \in X} \text{Tr}_k(A(x)).$$

Proof. For any $\epsilon > 0$, by continuity of Tr_k , there exists an integer n , $x_1, \dots, x_n \in X$, and a_1, \dots, a_n with $\sum_{i=1}^n a_i = 1$, such that

$$\text{Tr}_k\left(\sum_{i=1}^n a_i A(x_i)\right) < \text{Tr}_k\left(\int_X A(x) d\mu(x)\right) + \epsilon.$$

On the other hand, there exists $V_k \subset \mathbb{R}^n$ such that

$$\text{Tr}_k\left(\sum_{i=1}^n a_i A(x_i)\right) = \text{Tr}\left(\sum_{i=1}^n a_i A(x_i)\right) |_{V_k},$$

Furthermore,

$$\text{Tr}\left(\sum_{i=1}^n a_i A(x_i)\right) |_{V_k} = \sum_{i=1}^n a_i \text{Tr}(A(x_i) |_{V_k}) \geq \inf_{x \in X} \text{Tr}_k(A(x)).$$

This shows

$$\text{Tr}_k\left(\int_X A(x) d\mu(x)\right) + \epsilon > \inf_{x \in X} \text{Tr}_k(A(x)),$$

where ϵ can be arbitrarily small. Hence the proposition holds. \square

Definition 4. Given an n -dimensional Riemannian manifold M with curvature tensor R , for any $u \in T_x M$, we define a symmetric bilinear form $R_u(v_1, v_2) = -\langle R(u, v_1)u, v_2 \rangle$, where $v_1, v_2 \in T_x M$. In particular, if the manifold is non-positively curved, then R_u defines a positive semi-definite symmetric form on $T_x M$. Furthermore, we define the k -th Ricci curvature in direction u (compare with Definition 1) as

$$\text{Ric}_k(u) = -\text{Tr}_k(R_u).$$

Hence the n -th Ricci coincides with the standard Ricci curvature.

Lemma 5.1. *Let M be a closed manifold with non-positive curvature. Then the following conditions are equivalent.*

- (1) $\dim(\text{null}(R_v)) \leq n/4$ for all $v \in T^1 M$.

- (2) $\forall v \in T_x^1 M$, there exists a subspace $F_v \subset T_x M$ of dimension at least $3n/4$, such that $\langle v, F_v \rangle = 0$ and $R_v(u, u) \geq C_0$ for all $u \in F_v$, where C_0 is some universal constant that only depends on (M, g) .
- (3) $\forall v \in T_x^1 M$, the k -th eigenvalue (in increasing order) of R_v is at least C_0 when $k > n/4$, where C_0 is some universal constant that only depends on (M, g) .
- (4) There exists $\delta > 0$ that only depends on (M, g) , so that

$$\inf_{v \in T^1 M} \text{Tr}_k(R_v) \geq \delta$$

when $k > n/4$.

- (5) The manifold has strictly negative k -th Ricci when $k > n/4$. That is, $\text{Ric}_k(v) < 0$ for all $v \in T^1 M$ and $k > n/4$, or equivalently, $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} < 0$, as in Definition 1.

Proof. It is easy to see, by compactness of M , (4) and (5) are equivalent. We show by the loop of implications (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (4).

(4) \Rightarrow (3): Take V_k to be the span of the first k eigenvectors of R_v , with associated eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k$. By Definition 3, $\lambda_1 + \lambda_2 + \dots + \lambda_k = \text{Tr}(R_v|_{V_k}) \geq \text{Tr}_k(R_v) \geq \delta$, for some constant δ that only depends on (M, g) . So $\lambda_k \geq \delta/k \geq \delta/n$, with constant δ/n only depending on (M, g) .

(3) \Rightarrow (2): Take F_v to be the span of the last $n - k + 1$ eigenvectors (the $k, k + 1, \dots, n$ -th) of R_v , where k is selected to be $\lfloor \frac{n}{4} \rfloor + 1$. Then F_v is orthogonal to v as v is always in the null space of R_v which corresponds to the first eigenvalue 0. Note that $R_v(u) \geq \lambda_k \geq C_0$ for all $u \in F_v$, where λ_k is the k -th eigenvalue of R_v . Also note that the dimension of F_v is $n - k + 1 = n - \lfloor \frac{n}{4} \rfloor$, which is at least $3n/4$.

(2) \Rightarrow (1): For any $v \in T^1 M$, by the property of F_v , $F_v \cap \text{null}(R_v) = 0$. Therefore, $\dim(\text{null}(R_v)) + \dim(F_v) \leq n$, hence $\dim(\text{null}(R_v)) \leq n/4$.

(1) \Rightarrow (4): We set $l = \lfloor \frac{n}{4} \rfloor + 1$, and denote $\lambda_k(v)$ be the k -th eigenvalue of R_v . By (1), $\lambda_l(v) > 0$ for all $v \in T^1 M$. Since $\lambda_l(v)$ is continuous on v , and $T^1 M$ is compact, there exists a universal constant $\delta > 0$ such that $\lambda_l(v) \geq \delta$, hence for any $k > n/4$, we have

$$\inf_{v \in T^1 M} \text{Tr}_k(R_v) \geq \lambda_k(v) \geq \lambda_l(v) \geq \delta$$

□

Definition 5. We say a nonpositively curved manifold have *negative* $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature if it satisfies any of the five conditions above.

In order to obtain an estimate in Proposition 4.1, we will need to compare the eigenvalues of H_δ with K_δ . As an integrand of K_δ , the Hessian of Busemann functions are closely related to the sectional curvatures via the Jacobi equation. We recall that if X is a simply connected, nonpositively curved manifold, then the Busemann functions are of class C^2 (see [HIH77]), so that the Hessian DdB is a continuous symmetric form on X . And by

convexity, it is positive semi-definite everywhere. Moreover, we can bound below by sectional curvatures as follows.

Theorem 4. *Let \widetilde{M} be the universal cover of some closed non-positively curved manifold M , $x \in \widetilde{M}$ and $\theta \in \partial\widetilde{M}$. If $Y_0 \in T_x^1\widetilde{M}$ is any unit vector in the horocycle direction, that is, $Y \perp \gamma'_{x\theta}(0)$, where $\gamma_{x\theta}(t)$ is the geodesic ray connecting x and θ , then there exists a constant C that depends on the norm of up to the second derivative of curvature ($\|R\|$, $\|\nabla R\|$, and $\|\nabla^2 R\|$), such that*

$$DdB_{(x,\theta)}(Y_0, Y_0) \geq C(-\langle R(\gamma'_{x\theta}(0), Y_0)\gamma'_{x\theta}(0), Y_0 \rangle)^{3/2}$$

Proof. We extend Y_0 along the geodesic $\gamma_{x\theta}(t)$ to $Y(t)$, the unique stable Jacobi field with $Y(0) = Y_0$, then the Hessian $DdB_{(x,\theta)}(Y_0, Y_0)$ is the same as the second fundamental form in direction Y_0 of the horosphere determined by x and θ , which is further equal to $-\langle Y(0), Y'(0) \rangle$ (see [HIH77]). We now take the second covariant derivative along the geodesic ray of $\langle Y(t), Y(t) \rangle$, $\langle Y(t), Y(t) \rangle'' = 2(\langle Y'(t), Y'(t) \rangle + \langle Y(t), Y''(t) \rangle) = 2\|Y'(t)\|^2 + 2R_{\gamma'_{x\theta}(t)}(Y(t))$

Integrating along the geodesic ray, we obtain

$$2(\langle Y(\infty), Y'(\infty) \rangle - \langle Y(0), Y'(0) \rangle) = 2 \int_0^\infty (\|Y'(t)\|^2 + R_{\gamma'_{x\theta}(t)}(Y(t))) dt$$

Since $Y(t)$ is stable, $\|Y(t)\|^2$ converges to a constant, so its derivative $2\langle Y(t), Y'(t) \rangle$ goes to 0. Therefore, from the above equality, we obtain further that

$$\begin{aligned} DdB_{(x,\theta)}(Y_0, Y_0) &= -\langle Y(0), Y'(0) \rangle = \int_0^\infty (\|Y'(t)\|^2 + R_{\gamma'_{x\theta}(t)}(Y(t))) dt \\ &\geq \int_0^\infty R_{\gamma'_{x\theta}(t)}(Y(t)) dt \end{aligned}$$

To finish the proof of the theorem, we need the following lemma from calculus.

Lemma 5.2. *Let F be a C^2 function on $[0, \infty)$, if $F \geq 0$, and F'' is bounded by L , then there is a constant $C > 0$ that depends on L such that*

$$\int_0^\infty F(t) dt \geq C \cdot F(0)^{3/2}$$

Proof. First, we show $F(t) \geq (\sqrt{F(0)} - L't)^2$ on the interval $[0, \sqrt{F(0)}/L']$, for some L' depending only on L . If we denote $G(t) = F(t) - (\sqrt{F(0)} - L't)^2$, then $G(0) = 0$, and $G(\sqrt{F(0)}/L') = F(\sqrt{F(0)}/L') \geq 0$. Moreover, $G''(t) = F''(t) - 2L'^2$, so if we choose $L' > \sqrt{L/2}$, then $G''(t) < 0$. Therefore G is concave hence $G \geq 0$ on $[0, \sqrt{F(0)}/L']$. Using this result and note also that $F \geq 0$, we can estimate the integral

$$\int_0^\infty F(t) dt \geq \int_0^{\sqrt{F(0)}/L'} (\sqrt{F(0)} - L't)^2 dt = \frac{F(0)^{3/2}}{3L'} = C \cdot F(0)^{3/2}$$

where C is some constant depending on L . \square

We continue with the proof of Theorem 4. If we can apply the lemma above to the function $R_{\gamma'_{x\theta}(t)}(Y(t))$, then we get the inequality of the theorem. So it suffices to show that the second derivative of $R_{\gamma'_{x\theta}(t)}(Y(t))$ is bounded. Therefore we compute and estimate (by simply writing γ' , Y , and Y' for $\gamma'_{x\theta}(t)$, $Y(t)$, and $Y'(t)$):

$$\begin{aligned} |(R_{\gamma'_{x\theta}(t)}(Y(t)))''| &= |(\langle (\nabla R)_{\gamma'}(\gamma', Y)\gamma', Y \rangle + 2\langle R(\gamma', Y)\gamma', Y' \rangle)'| \\ &= |(\langle (\nabla^2 R)_{(\gamma', \gamma')}(\gamma', Y)\gamma', Y \rangle + 4\langle (\nabla R)_{\gamma'}(\gamma', Y)\gamma', Y' \rangle \\ &\quad + 2\langle R(\gamma', Y')\gamma', Y' \rangle + 2\langle R(\gamma', Y)\gamma', Y'' \rangle)| \\ &\leq C(\|Y\|^2 + \|Y'\|^2) \end{aligned}$$

where the last inequality uses the Jacobi equation $Y'' = -R(\gamma', Y)\gamma'$, and C is a constant depending on $\|R\|$, $\|\nabla R\|$ and $\|\nabla^2 R\|$. We also note that since Y is the stable Jacobi field, $\|Y(t)\|$ is non-increasing along the geodesic ray and therefore $\|Y(t)\| \leq \|Y(0)\| = 1$. Thus we only need to show $\|Y'\|^2$ is bounded. However, we have

$$\begin{aligned} |\langle Y'(t), Y'(t) \rangle'| &= |2\langle Y'', Y' \rangle| = 2|\langle R(\gamma', Y)\gamma', Y' \rangle| \\ &\leq C_1 \sqrt{\langle R(\gamma', Y)\gamma', Y \rangle \langle R(\gamma', Y')\gamma', Y' \rangle} \\ &\leq C_2 \sqrt{\langle -R(\gamma', Y)\gamma', Y \rangle} \|Y'\| \\ &\leq C_2 (\langle -R(\gamma', Y)\gamma', Y \rangle + \|Y'\|^2) \\ &= C_2 \langle Y, Y' \rangle' \end{aligned}$$

where the first inequality is due to the fact that $-R$ is positive semi-definite, the second uses the bound of $\|R\|$, and the third inequality is the Cauchy-Schwarz inequality. Here C_1 is some universal constant, and C_2 is a constant depending on $\|R\|$. Integrating the above inequality, we obtain, for any $0 < t < s < \infty$,

$$|\langle Y'(t), Y'(t) \rangle - \langle Y'(s), Y'(s) \rangle| \leq C_2 |\langle Y(t), Y'(t) \rangle - \langle Y(s), Y'(s) \rangle|$$

As we see earlier in the proof, $\langle Y(s), Y'(s) \rangle$ increases to 0 as s approaches to ∞ . Note that

$$\int_0^\infty \|Y'(t)\|^2 dt \leq \int_0^\infty \|Y'(t)\|^2 + R_{\gamma'_{x\theta}(t)}(Y(t)) dt < \infty$$

so $\langle Y'(s), Y'(s) \rangle$ also goes to 0 as s approaches to ∞ . So by taking $s \rightarrow \infty$, we have

$$\|Y'(t)\|^2 \leq C_2 |\langle Y(t), Y'(t) \rangle| \leq -C_2 \langle Y(0), Y'(0) \rangle = C_2 DdB_{(x,\theta)}(Y_0, Y_0)$$

But by the comparison theorem the Hessian $DdB_{(x,\theta)}(Y_0, Y_0)$ is bounded above by some constant depending on $\|R\|$. This shows $\|Y'\|$ is bounded by some constant on $\|R\|$, hence the second derivative of $R_{\gamma'_{x\theta}(t)}(Y(t))$ is

bounded by some constant on $\|R\|$, $\|\nabla R\|$, and $\|\nabla^2 R\|$, and in view of Lemma 5.2, we obtain the inequality of the theorem. \square

Corollary 1. *Under the assumption of Theorem 4, if M has negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature, then*

$$\text{Tr}_{k+1}(DdB_{(x,\theta)}(\cdot, \cdot)) \geq C_0$$

where $k = \lfloor \frac{n}{4} \rfloor$ and C_0 depends on the negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci constant in Lemma 5.1, in particular it depends on (M, g) .

Proof. We choose an orthonormal frame e_1, e_2, \dots, e_{k+1} of the $k+1$ least eigenvectors of $DdB_{(x,\theta)}(\cdot, \cdot)$, so that $\text{Tr}_{k+1}(DdB_{(x,\theta)}(\cdot, \cdot)) = \sum_{i=1}^{k+1} DdB_{(x,\theta)}(e_i, e_i)$. According to Theorem 4 and Hölder's inequality, we obtain

$$\sum_{i=1}^{k+1} DdB_{(x,\theta)}(e_i, e_i) \geq C \sum_{i=1}^{k+1} R_{v_{x\theta}}(e_i, e_i)^{3/2} \geq C' \left(\sum_{i=1}^{k+1} R_{v_{x\theta}}(e_i, e_i) \right)^{3/2}$$

The negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature condition implies

$$\sum_{i=1}^{k+1} R_{v_{x\theta}}(e_i, e_i) \geq \text{Tr}_{k+1}(R_{v_{x\theta}}) \geq C''$$

for some constant C'' depending on (M, g) . Combining the above two inequalities gives the result of this corollary. \square

As a first step towards Theorem 5, we use the result of Theorem 4 to obtain the following lemma, just to compare pointwisely the integrands of H_δ and K_δ —the two positive semi-definite forms described in Section 4. We remark that the power $2/3$ in the following lemma (which actually traces back to lemma 5.2) directly leads to our imposed “ $n/4$ ” condition. If this power can be improved to be closer to 1, then the resulting k -Ricci condition could also be slightly weakened, but is still limited to an “ $n/3$ ” condition.

Lemma 5.3. *Suppose M is a closed non-positively curved manifold with negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature, and \tilde{M} is its Riemannian universal cover. Let $x \in \tilde{M}$, and $\theta \in \partial \tilde{M}$. Then there is a constant C that depends on (M, g) , such that for all $v \in T_x^1 M$ and all $u \in F_v$ (where F_v satisfies (2) of Lemma 5.1), we have*

$$dB_{(x,\theta)}^2(u, u) \leq C(DdB_{(x,\theta)}(v, v))^{2/3}$$

Proof. We decompose v as $v_1 + v_2$ where v_1 is in the direction of $v_{x\theta}$, and v_2 is in the orthogonal direction, and we denote α the angle between $v_{x\theta}$ and

v . So by Theorem 4, we can estimate

$$\begin{aligned} DdB_{(x,\theta)}(v, v) &= DdB_{(x,\theta)}(v_2, v_2) \geq (\sin^2 \alpha)(C \cdot R_{v_{x\theta}}(\frac{v_2}{\|v_2\|}, \frac{v_2}{\|v_2\|}))^{3/2} \\ &= \frac{C^{3/2}}{|\sin \alpha|} R_{v_{x\theta}}(v_2, v_2)^{3/2} \geq C^{3/2} R_{v_{x\theta}}(v, v)^{3/2} \\ &= C^{3/2} R_v(v_{x\theta}, v_{x\theta})^{3/2} \end{aligned}$$

Note that in the above estimate, if $\sin \alpha = 0$, then v and $v_{x\theta}$ are parallel, and $dB_{(x,\theta)}^2(u, u) = 0$, the inequality holds automatically. We also note that when restricted to the subspace F_v , R_v have eigenvalues at least C_0 according to Lemma 5.1, hence

$$R_v(v_{x\theta}, v_{x\theta}) \geq C_0 \cos^2(\angle(v_{x\theta}, F_v)) \geq C_0 \cos^2(\angle(v_{x\theta}, u)) = C_0 dB_{(x,\theta)}^2(u, u),$$

Combining the two inequalities, we obtain

$$dB_{(x,\theta)}^2(u, u) \leq C(DdB_{(x,\theta)}(v, v))^{2/3}$$

for some constant C that depends on $\|R\|$, $\|\nabla R\|$, $\|\nabla^2 R\|$, and the Ricci constant C_0 of Lemma 5.1, in particular it depends on (M, g) . This completes the proof. \square

Theorem 5. *Suppose M is a closed non-positively curved manifold with negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature, and \widetilde{M} is its Riemannian universal cover. Let $x \in \widetilde{M}$, $\theta \in \partial \widetilde{M}$, and ν be any probability measure that has full support on $\partial \widetilde{M}$. Then there exists a universal constant C that only depends on (M, g) , so that*

$$\frac{\det(\int_{\partial \widetilde{M}} d\widetilde{B}_{(x,\theta)}^2(\cdot, \cdot) d\nu(\theta))^{1/2}}{\det(\int_{\partial \widetilde{M}} DdB_{(x,\theta)}(\cdot, \cdot) d\nu(\theta))} \leq C$$

Proof. We follow the framework of [BCG95], and set $K_{x,\theta} := DdB_{(x,\theta)}(\cdot, \cdot)$, $H_{x,\theta} := dB_{(x,\theta)}^2(\cdot, \cdot)$, and $K := \int_{\partial \widetilde{M}} K_{x,\theta}(\cdot, \cdot) d\nu(\theta)$, $H = \int_{\partial \widetilde{M}} H_{x,\theta}(\cdot, \cdot) d\nu(\theta)$. Let (λ_1, v) be a smallest eigenvalue (vector) of K . Then there is a constant C' depending on (M, g) such that, for any $u \in F_v$, we have

$$\begin{aligned} H(u, u) &= \int_{\partial \widetilde{M}} H_{x,\theta}(u, u) d\nu(\theta) \leq C' \int_{\partial \widetilde{M}} K_{x,\theta}(v, v)^{2/3} d\nu(\theta) \\ &\leq C' (\int_{\partial \widetilde{M}} K_{x,\theta}(v, v) d\nu(\theta))^{2/3} = C' \lambda_1^{2/3} \end{aligned}$$

where the first inequality is due to Lemma 5.3, and the second is the Hölder inequality. Therefore we can find an orthonormal frame e_1, e_2, \dots, e_{n-k} at x so that $H(e_i, e_i) \leq C' \lambda_1^{2/3}$ for $1 \leq i \leq n - k$, where $k = \lfloor \frac{n}{4} \rfloor$. This implies that

$$\text{Tr}_{n-k}(H) \leq \sum_{i=1}^{n-k} H(e_i, e_i) \leq (n - k) C' \lambda_1^{2/3}.$$

If we further denote $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ the eigenvalues of H , then, we have

$$\mu_i \leq \text{Tr}_{n-k}(H) \leq (n-k)C'\lambda_1^{2/3},$$

for $1 \leq i \leq n-k$, since $\text{Tr}_{n-k}(H)$ is the sum of $n-k$ least eigenvalues of H . Note also that the eigenvalues of H are at most 1 and $k \leq n/4$, so we can estimate the following

$$\frac{(\det H)^{1/2}}{\det K} = \frac{\prod_{i=1}^n \mu_i^{1/2}}{\prod_{i=1}^n \lambda_i} \leq \frac{[(n-k)C'\lambda_1^{2/3}]^{\frac{n-k}{2}}}{\lambda_1^k \lambda_{k+1}^{n-k}} \leq \frac{C''}{\lambda_{k+1}^{n-k}}$$

for some constant C'' depending on (M, g) . Finally, we can bound λ_{k+1} in the following way:

$$\lambda_{k+1} \geq \frac{1}{k+1} \text{Tr}_{k+1}(K) \geq \frac{1}{k+1} \inf_{\theta \in \partial \tilde{M}} \text{Tr}_{k+1}(K_{x,\theta}) \geq \frac{C_0}{k+1},$$

where the second inequality is due to Proposition 5.1, and the third inequality is by Corollary 1. Therefore, we conclude by combining the above inequalities that

$$\frac{(\det H)^{1/2}}{\det K} \leq C'' \left(\frac{k+1}{C_0} \right)^{n-k} \leq C$$

where C is a constant depending on (M, g) , or more specifically, C depends on the dimension n , $\|R\|$, $\|\nabla R\|$, $\|\nabla^2 R\|$, and the corresponding Ricci constant C_0 of Lemma 5.1 \square

As a direct consequence, we prove Theorem 1.

Proof of Theorem 1: According to Theorem 5 and Proposition 4.1, we see that the barycentric straightening is a straightening on \tilde{M} , if M satisfies the negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature condition. Therefore, by Theorem 3, the simplicial volume $\|M\| > 0$.

6. THE NATURAL MAPS

In this section we describe the natural map which is an essential ingredient in the method of Besson-Courtois-Gallot to prove the volume estimate of Theorem 2.

Let M continue to denote a closed oriented nonpositively curved Riemannian manifold with negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature, and let N be an arbitrary closed oriented manifold of the same dimension. Assume f is a continuous map from N to M . Let ϕ denote a choice of lift of f to universal covers, i.e. $\phi = \tilde{f} : \tilde{N} \rightarrow \tilde{M}$. We will also denote the metric and Riemannian volume form on universal cover \tilde{N} by g and dg respectively. Then for each $s > h(g)$ and $y \in \tilde{N}$ consider the probability measure μ_y^s on \tilde{N} in the Lebesgue class with density given by

$$\frac{d\mu_y^s}{dg}(z) = \frac{e^{-sd(y,z)}}{\int_N e^{-sd(y,z)} dg}.$$

The μ_y^s are well defined by the choice of s .

Consider the push-forward $\phi_*\mu_y^s$, which is a measure on \widetilde{M} . Define σ_y^s to be the convolution of $\phi_*\mu_y^s$ with the Patterson-Sullivan measure ν_z on $\partial\widetilde{M}$.

In other words, for $U \subset \partial\widetilde{M}$ a Borel set, define

$$\sigma_y^s(U) = \int_{\widetilde{M}} \nu_z(U) d(\phi_*\mu_y^s)(z)$$

For each $s > h(M)$, σ_y^s has full support since each ν_z does. Therefore we define maps $\widetilde{F}_s : \widetilde{N} \rightarrow \widetilde{M}$ by

$$\widetilde{F}_s(y) = \text{bar}(\sigma_y^s).$$

Note that σ_y^s may not be a probability measure since we do not necessarily have $\|\nu_z\| = 1$. However the barycenter is invariant under scaling of the measure.

The equivariance of ϕ and of $\{\nu_z\}$ and μ_y^s implies that \widetilde{F}_s is also equivariant. Hence \widetilde{F}_s descends to a map $F_s : N \rightarrow M$. It is easy to see that F_s is homotopic to f .

Proposition 6.1. *The map $\Psi_s : [0, 1] \times N \rightarrow M$ defined by*

$$\Psi_s(t, y) = F_{s + \frac{t}{1-t}}(y)$$

is a homotopy between $\Psi_s(0, \cdot) = F_s$ and $\Psi_s(1, \cdot) = f$.

Proof. From its definitions, $\widetilde{F}_s(y)$ is continuous in s and y . Observe that for fixed y , $\lim_{s \rightarrow \infty} \sigma_y^s = \nu_{\phi(y)}$. It follows that $\lim_{s \rightarrow \infty} \widetilde{F}_s(y) = \phi(y)$. This implies the proposition. \square

As in [BCG95], the implicit function theorem together with Lemma 3.1 implies that F_s is C^1 , and we will estimate its Jacobian.

Theorem 6 (The Jacobian Estimate). *For all $s > h(g)$ and all $y \in N$ we have*

$$|\text{Jac } F_s(y)| \leq C s^n$$

for some constant C , depending only on \widetilde{M} .

Proof. We obtain the differential of F_s by implicit differentiation:

$$0 = D_{x=F_s(y)} \mathcal{B}_{\sigma_y^s}(x) = \int_{\partial\widetilde{M}} dB_{(F_s(y), \theta)}(\cdot) d\sigma_y^s(\theta)$$

Hence as 2-forms

$$\begin{aligned} 0 &= D_y D_{x=F_s(y)} \mathcal{B}_{\sigma_y^s}(x) = \int_{\partial\widetilde{M}} D dB_{(F_s(y), \theta)}(D_y F_s(\cdot), \cdot) d\sigma_y^s(\theta) \\ &\quad - s \int_N \int_{\partial\widetilde{M}} dB_{(F_s(y), \theta)}(\cdot) \langle \nabla_y d(y, z), \cdot \rangle d\nu_{\phi(z)}(\theta) d\mu_y^s(z) \end{aligned}$$

The distance function $d(y, z)$ is Lipschitz and C^1 off of the cut locus which has Lebesgue measure 0. It follows from the Implicit Function Theorem (see

[BCG98]) that F_s is C^1 for $s > h(g)$. By the chain rule,

$$\text{Jac } F_s = s^n \frac{\det \left(\int_N \int_{\partial \widetilde{M}} dB_{(F_s(y), \theta)}(\cdot) \langle \nabla_y d(y, z), \cdot \rangle d\nu_{\phi(z)}(\theta) d\mu_y^s(z) \right)}{\det \left(\int_{\partial \widetilde{M}} DdB_{(F_s(y), \theta)}(\cdot, \cdot) d\sigma_y^s(\theta) \right)}$$

Dividing numerator and denominator by $\frac{1}{\|\sigma_y^s\|}$ and applying Hölder's inequality to the numerator gives:

$$|\text{Jac } F_s| \leq s^n \frac{\det \left(\frac{1}{\|\sigma_y^s\|} \int_{\partial \widetilde{M}} dB_{(F_s(y), \theta)}^2 d\sigma_y^s(\theta) \right)^{1/2} \det \left(\int_N \langle \nabla_y d(y, z), \cdot \rangle^2 d\mu_y^s(z) \right)^{1/2}}{\det \left(\frac{1}{\|\sigma_y^s\|} \int_{\partial \widetilde{M}} DdB_{(F_s(y), \theta)}(\cdot, \cdot) d\sigma_y^s(\theta) \right)}$$

Using that $\text{trace } \langle \nabla_y d(y, z), \cdot \rangle^2 = |\nabla_y d(y, z)|^2 = 1$, except possibly on a measure 0 set, we may estimate

$$\det \left(\int_N \langle \nabla_y d(y, z), \cdot \rangle^2 d\mu_y^s(z) \right)^{1/2} \leq \left(\frac{1}{\sqrt{n}} \right)^n$$

Therefore

$$(6.1) \quad |\text{Jac } F_s| \leq \left(\frac{s}{\sqrt{n}} \right)^n \frac{\det \left(\frac{1}{\|\sigma_y^s\|} \int_{\partial \widetilde{M}} dB_{(F_s(y), \theta)}^2 d\sigma_y^s(\theta) \right)^{1/2}}{\det \left(\frac{1}{\|\sigma_y^s\|} \int_{\partial \widetilde{M}} DdB_{(F_s(y), \theta)}(\cdot, \cdot) d\sigma_y^s(\theta) \right)}$$

Applying Theorem 5 completes the proof. \square

Now we prove Theorem 2, which we restate for convenience:

Theorem 2. *Given any closed Riemannian manifolds N and M of dimension n , and assume M is nonpositively curved and has $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} < 0$. There is a constant $C > 0$, depending only on the metric of \widetilde{M} and scale invariant, such that if $f : N \rightarrow M$ is any continuous map, then*

$$h(N)^n \text{Vol}(N) \geq C |\deg f| h(M)^n \text{Vol}(M)$$

where $h(N)$ is the volume growth entropy of N .

Proof. From degree theory note that f is isotopic to a C^1 -map, which we again denote by f , and we have

$$\begin{aligned} |\deg(f)| \text{Vol}(M) &= \left| \int_N f^* dg_0 \right| = \left| \int_N F_s^* dg_0 \right| \\ &\leq \int_N |\text{Jac } F_s| dg \leq C_M s^n \text{Vol}(N). \end{aligned}$$

The second equality follows from Proposition 6.1 and the last inequality from Theorem 6. Finally we take $C_1 = \frac{1}{C_M}$ and then take the limit as $s \rightarrow h(N)$ to obtain $h(N)^n \text{Vol}(N) \geq C_1 |\deg f| \text{Vol}(M)$. Applying this inequality to

the identity map and $N = M$ shows that $h(M) > 0$. Hence we may take $C = \frac{C_1}{h(M)^n}$ to obtain the inequality of the theorem.

From the proof of the Jacobian estimate, we observe that C_M only depends on the metric \tilde{g} on \tilde{M} , which is diffeomorphic to \mathbb{R}^n . The same dependence is well known for $h(M)$. Also, the quantity $h(M)^n \text{Vol}(M)$ is scale invariant, and therefore if C is not scale invariant we may choose it to be so by taking the supremum of its values over scalings of M . \square

Remark 6.1. Observe that the above theorem implies that for any metric g_0 on M , $h(M, g_0) \text{Vol}(M, g_0) \geq \sup_g C(g) h(M, g_0) \text{Vol}(M, g_0) > 0$. Hence if $C = C(M, g)$ cannot be chosen to be a topological invariant of M , then there must be a sequence of nonpositively curved metrics (M, g_i) with $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1}(g_i) < 0$ such that $h(M, g_i) \text{Vol}(M, g_i) \rightarrow \infty$.

7. APPLICATIONS

In this section we provide some applications of the above theorems. First we remark that Gromov in [Gro82] showed a proportionality principle for manifolds M and N sharing a common universal cover. Namely,

$$\frac{\|N\|}{\text{Vol}(N)} = \frac{\|M\|}{\text{Vol}(M)}.$$

More generally, if $f : N \rightarrow M$ is any map of nonzero degree then since f induces a map at the level of singular chains such that $f_*[N] = \deg(f)[M]$, we have that $\|M\| > 0$ implies $\|N\| > 0$.

Now we recall that the dual of equivariant chains on \tilde{M} with L^1 coefficients are equivariant cochains on \tilde{M} with L^∞ coefficients. This leads to the following application.

7.1. Bounded cohomology. The notion of bounded cohomology is defined in [Gro82], and has been studied in various contexts. If M is a closed manifold, then the positivity of the simplicial volume $\|M\|$ is equivalent to the surjectivity of the top dimensional comparison map $\eta : H_b^n(M, \mathbb{R}) \rightarrow H^n(M, \mathbb{R})$ [Gro82]. Hence our Theorem 1 immediately implies the following:

Corollary 2. *If M is an n -dimensional closed nonpositively curved manifold that has $\text{Ric}_{\lfloor \frac{n}{4} \rfloor + 1} < 0$, then the comparison map*

$$\eta : H_b^n(M, \mathbb{R}) \rightarrow H^n(M, \mathbb{R})$$

is surjective. In particular, the bounded cohomology $H_b^n(M, \mathbb{R})$ is nontrivial.

One can then naturally ask: are there surjectivity results in lower degrees? The question is already quite interesting when M is locally symmetric of higher rank, as a positive answer leads to showing Dupont's Conjecture [Dup79], which states that the comparison maps from the continuous bounded cohomology of the corresponding semisimple Lie group into the standard continuous cohomology is always surjective. Recently, the same

method of barycentric straightening has been applied by Lafont and the second author [LW15] in symmetric spaces, to answer Dupont's Conjecture in high degrees. Showing surjectivity in dimension p requires a uniform bound on the p -Jacobian of barycentrically straightened simplices.

The same result holds in our context as well.

Corollary 3. *If M has $\text{Ric}_{k+1} < 0$ for some $k \leq \lfloor \frac{n}{4} \rfloor$, then the comparison map $\eta : H_b^*(M, \mathbb{R}) \rightarrow H^*(M, \mathbb{R})$ is surjective when $* \geq 4k$.*

Proof. The proof of Theorem 5 already works verbatim for the case that, for the same k in the proof, $k \leq \lfloor \frac{n}{4} \rfloor$. This implies that we obtain the same bounds on the determinant restricted to $4k$ -dimensional subspaces. Consequently, the same proof yields a bound for the p -Jacobians of the straightening map for $p \geq 4k$.

An essentially identical argument to that of [LW15] can then be applied in our context where M is geometric rank one to show surjectivity of the comparison maps in the degrees at least $4k$. \square

7.2. The co-Hopf property. We recall the following.

Definition 6. A group G has the *co-Hopf property*, in which case G is called *co-Hopfian*, if every injective endomorphism $h : G \rightarrow G$ is also surjective.

Corollary 4. *Let N be any closed oriented manifold admitting a nonzero degree map to a closed oriented nonpositively curved manifold M with negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature. Then $\pi_1(N)$ is co-Hopfian.*

Proof. Suppose not, then there is an injective endomorphism $g_* : \pi_1(N) \rightarrow \pi_1(N)$ with proper image. Consider the covering map $g : N \rightarrow N$ corresponding to the subgroup $g_*\pi_1(N) < \pi_1(N)$. Since N is closed $\deg(g)$ is finite and it divides the index $[\pi_1(N) : g_*\pi_1(N)]$. Hence $|\deg(g)| > 1$. Since N admits a map $f : N \rightarrow M$ of nonzero degree, for any $k \in \mathbb{N}$ we have a map $f \circ g^k : N \rightarrow M$ with $|\deg(f \circ g^k)| = |\deg(f) \deg(g)^k| \geq 2^k$, which contradicts Theorem 2 for sufficiently large k . \square

Note that the above corollary applies to the case of $N = M$ as well using the identity map.

7.3. The Minvol and Minent invariants. The *minimal volume* $\text{Minvol}(M)$ and *lower Ricci minimal volume* $\text{Minvol}_{\text{Ric}}(M)$ are closely related smooth topological invariants of M defined as,

$$\text{Minvol}(M) = \inf \{ \text{Vol}(M, g) \mid -1 \leq K_g \leq 1 \}$$

and

$$\text{Minvol}_{\text{Ric}}(M) = \inf \{ \text{Vol}(M, g) \mid -(n-1) \leq \text{Ric}_g \},$$

where the infimums are over all C^∞ Riemannian metrics g on M and K_g (resp. Ric_g) represents the sectional (resp. Ricci) curvatures of g .

A similar invariant is the *minimal (volume growth) entropy* invariant,

$$\text{Minent}(M) = \inf \{ h(M, g) \mid \text{Vol}(M, g) = 1 \}$$

Note that since $h(M, g)^n \text{Vol}(M, g)$ is a scale invariant quantity,

$$\text{Minent}(M) = \inf_g \left\{ h(M, g) \text{Vol}(M, g)^{\frac{1}{n}} \right\} = (n-1) \inf \left\{ \text{Vol}(M, g)^{\frac{1}{n}} \mid h(M, g) \leq n-1 \right\}.$$

The Bishop volume comparison Theorem (see [CRD84]) implies that $h(g) \leq n-1$ whenever $-(n-1) \leq \text{Ric}_g$. The latter condition occurs whenever $-1 \leq K_g$, and consequently we have the following relationships:

$$\text{Minvol}(M) \geq \text{Minvol}_{\text{Ric}}(M) \geq \left(\frac{\text{Minent}(M)}{n-1} \right)^n.$$

Theorem 2 then directly implies the following.

Corollary 5. *Let N be any closed oriented manifold admitting a nonzero degree map to a closed oriented nonpositively curved manifold M with negative $(\lfloor \frac{n}{4} \rfloor + 1)$ -Ricci curvature. Then $\text{Minvol}(N)$, $\text{Minvol}_{\text{Ric}}(N)$ and $\text{Minent}(N)$ are all positive.*

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INDIANA UNIVERSITY

E-mail address: `connell@indiana.edu`

INDIANA UNIVERSITY

E-mail address: `wang679@iu.edu`